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LITERATURE CITED

1. S. Z. Belen'kii and E. S. Fradkin, "The theory of turbulent mixing," Tr. Fiz. Inst. Akad. Nauk SSSR im. P. N. Lebedeva, 29, 207 (1965).
2. N. L. Krasheninnikova, "The not fully established motion of a gas displaced by a piston," Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk, No. 8 (1955).
3. V. B. Adamskii and N. A. Popov, "The motion of a gas under the action of pressure at a piston, varying according to a power law," Prikl. Mat. Mekh., 23, No. 3, 564 (1959).
4. V. E. Neuvazhaev, "Nonadiabatic motions in an ideal gas (self-similar solutions)," Tr. Mat. Inst. Akad. Nauk SSSR, 22, 24 (1973).
5. V. E. Neuvazhaev and V. G. Yakovlev, "Turbulent mixing of an interface in a numerical gasdynamic calculation," Zh. Vychisl. Mat. Mat. Fiz., No. 2 (1976).
6. L. I. Sedov, Similarity and Dimensional Methods in Mechanics, Academic Press (1959).

STEADY-STATE PERTURBATIONS IN A LIQUID CONTAINING GAS BUBBLES

V. V. Goncharov, K. A. Naugol'nykh, and S. A. Rybak

UDC 532.529.5

The problem of wave propagation in a liquid with gas bubbles, which is an example of a nonlinear dispersive medium, is usually treated in the approximation of weak nonlinearity and dispersion [1, 2], and a solution has been successfully obtained only in some special cases corresponding to a strong variation of the bubble radius [3]. In contrast to this, it is shown in this paper that a wider class of solutions is successfully found for stationary waves which correspond to highly nonlinear pulsations of the bubbles. At the same time, periodic solutions appear along with solutions of the soliton type, which correspond to the situation in which nonlinear and dispersive effects just compensate each other.

One-dimensional acoustic waves in a bubble medium can be described by a system of linear acoustic equations which take account of the presence of gas bubbles:

$$\partial\rho/\partial t + \rho_0\partial v/\partial x = 0;\ \partial v/\partial t + (1/\rho_0)\partial p/\partial x = 0;\ \rho/\rho_0 = \left[(1-z)/\rho_0 c_0^2\right] p - nV \tag{1}$$

and by the Rayleigh nonlinear equation for oscillations of a gas bubble

$$Rd^2R/dt^2 + (3/2)(dR/dt)^2 = (p_0/\rho_0)[(R_0/R)^{3\gamma} - 1] - p/\rho_0. \tag{2}$$

Since $p = p(t, x)$, then $R = R(t, x)$ and $dR/dt \approx \partial R/\partial t$ on the condition that one can neglect the convective nonlinear terms. Here ρ_0, p_0, and c_0 are the equilibrium values of the density, pressure, and speed of sound, respectively, in a liquid without bubbles; R_0 is the equilibrium bubble radius; R is its instantaneous radius; γ is the adiabatic exponent for the gas in the bubble; n is the number of bubbles per unit volume; z is the bubble concentration; and ρ, p, v, and V are the variations in the density, pressure, speed of the liquid's particles, and the bubble volume, respectively.

If one introduces the equilibrium bubble volume $V_0 = (4/3)\pi R_0^3$ $[z = nV_0, V_0 + V = (4/3)\pi R^3]$ the eigenfrequency of the oscillations of the bubble $\omega_0^2 = 3\gamma p_0/\rho_0 R_0^2$, and also the dimen-

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$$X = p/\rho_0 c_0^2, \quad Y = \frac{V}{V_0}, \quad \tau = \omega_0 t,$$
$$\xi = \frac{\omega_0}{c_0} x, \quad \bar{\rho} = \rho/\rho_0, \quad \bar{v} = v/c_0;$$

then the system of equations (1) and (2) is brought to the dimensionless form

$$\partial\bar{\rho}/\partial\tau + \partial\bar{v}/\partial\xi = 0, \ \partial\bar{v}/\partial\tau + \partial X/\partial\xi = 0, \ \bar{\rho} = (1 - z)X - zY,$$
$$\partial^2 Y/\partial\tau^2 = (1/6)(1 + Y)^{-1}(\partial Y/\partial\tau)^2 + (1 + Y)^{1/3}\{[(1 + Y)^{-\gamma} - 1]/\gamma - \delta X\}. \tag{3}$$

Here the parameter $\delta = \rho_0 c_0^2/\gamma p_0 = \rho_0 c_0^2/\rho_B c_B^2 \gg 1$, where c_B and ρ_B are the equilibrium speed of sound and the density of the gas in the bubble.

We will further consider only steady-state solutions of the system (3) of the form $Y = Y(\eta)$, where $\eta = \xi - c\tau$ and c is some constant equal to the speed (in units of c_0) of movement of the unknown perturbation in the medium. For steady-state solutions, the coupling equations

$$X = X(\eta) = X_0 + SY, \ \bar{v} = \bar{v}(\eta) = \bar{v}_0 + (S/c)Y,$$
$$\bar{\rho} = \bar{\rho}(\eta) = (1 - z)X_0 + (S/c^2)Y, \tag{4}$$

where $S = zc^2/[(1 - z)c^2 - 1]$, follow from the system (3); the constants X_0 and $\bar{v}_0$ can be set to zero, thereby having included them in the equilibrium values p_0, ρ_0, and v_0. In this case, the last equation of (3) is reduced to an ordinary differential equation with respect to Y

$$d^2Y/d\eta^2 - (1/6)(1 + Y)^{-1}(dY/d\eta)^2 + [(1 + Y)^{1/3}/c^2]\{[(1 + Y)^{-\gamma} - 1]/\gamma - DY\}, \tag{5}$$

where

$$D = S\delta = \frac{c^2}{c_2^2}\frac{c_1^2 - c_2^2}{c^2 - c_1^2} = D - 1, \qquad D = \frac{c_1^2}{c_2^2}\frac{c^2 - c_2^2}{c^2 - c_1^2},$$
$$c_1^2 = (1 - z)^{-1}, \quad c_2^2 = (1 - z + z\delta)^{-1}.$$

We note that $c_2^2 < c_1^2$ follows from the conditions $z\delta > 0$ and $z < 1$.

The bounded solutions of Eq. (5) for some fixed value of the speed c are cited for perturbations propagating in a liquid without change of shape. One should bear in mind that Eq. (5) is derived from the linear acoustic equations (1) (only the nonlinearity of the oscillations of the bubbles was taken into account). Consequently, the condition of the smallness of the Mach acoustic number $M = \max|\bar{v}|/c \ll 1$, should be fulfilled, which, along with (4), gives a constraint on the maximum value of Y:

$$\frac{zc_1^2 \max|Y|}{|c^2 - c_1^2|} \ll 1. \tag{6}$$

In the linear approximation the solution of Eq. (5) is of the form

$$Y = Ae^{i\varkappa\eta} = Ae^{i(\varkappa\xi - f\tau)}, \ \varkappa = D/c^2, \ f = c\varkappa.$$

The requirement of the boundedness of Y reduces to the condition $\varkappa^2 \geqslant 0$, whence we obtain two regions of permissible values of the speed c: $c^2 > c_1^2$ and $c^2 < c_2^2$. The dispersion equation of linear acoustic waves in a bubble medium is written in the form $f = \sqrt{D} = \frac{c_1}{c_2}\sqrt{\frac{c^2 - c_2^2}{c^2 - c_1^2}}$.

The corresponding dispersion curves in the coordinates c and f are shown in Fig. 1. It is obvious that the speed c_1 corresponds to the speed of high-frequency waves $f \gg 1 (\omega \gg \omega_0)$; in this connection, the oscillations of the bubbles are not expressed in the propagation of waves [the term zY in the third equation of (3) is omitted]. The presence of bubbles is exhibited only in the variation of the equilibrium of the liquid $\rho_{eff} = \rho_0(1 - z)$. The speed c_2 corresponds to low-frequency waves $f \ll 1$.

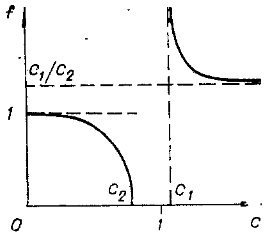

Fig. 1

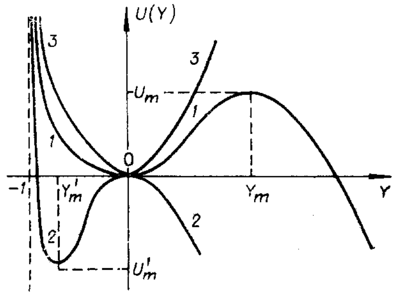

Fig. 2

In order to find a bounded solution of the nonlinear Eq. (5), let us multiply the latter by $2(1 + Y)^{-1/3}dY/d\eta$ to obtain

$$\frac{d}{d\eta}\left\{(1+Y)^{-1/3}\left(\frac{dY}{d\eta}\right)^2+\int\limits_0^Y \frac{2}{c^2}\left[Dy-\frac{(1+y)^{-\gamma}-1}{\gamma}\right]dy\right\}=0,$$

whence the first integral of Eq. (5) follows directly:

$$(1+Y)^{-1/3}(dY/d\eta)^2+\beta^2 U(Y)=\beta^2 H, \tag{7}$$

where

$$\beta=\sqrt{(1+\gamma)/3c^2};\ \alpha=(1/2)\gamma(\gamma-1)D;$$
$$U(Y)=[6/\gamma(\gamma^2-1)][(1+Y)^{-(\gamma-1)}-1+(\gamma-1)Y+\alpha Y^2].$$

Equation (7) has the meaning of an energy conservation law. Actually, transforming to dimensional variables, we obtain

$$(1+Y)^{-1/3}\left(\frac{dY}{d\eta}\right)^2=\frac{18\omega_0^2}{c^2V_0\rho_0}T_k,$$

where $T_k = \rho_0 (V_0 + V)(dR/dt)^2$ is the kinetic energy of an individual bubble in the liquid. Then the quantity U(Y) is proportional to the potential energy, and the constant H is proportional to the conserved total energy of the bubble. In the quadratic approximation

$$U(Y)=Y^3-[3/(1+\gamma)]DY^2,\ |Y|\ll 1. \tag{8}$$

The general solution of Eq. (7) is written in the implicit form

$$\beta\eta+\Theta=\pm\int\limits_{Y_1}^{Y}(1+y)^{-1/6}(H-U(y)^{-1/2}dy.$$

Here Y can take the values $Y_1 \leqslant Y \leqslant Y_2$, where Y_1 and Y_2 are two finite roots of the function $H - U(Y)$, between which $H \geqslant U(Y)$. The solution $Y = Y(\beta\eta + \Theta$ is periodic (a soliton is obtained in the limiting case of a multiple root Y_1 or Y_2) with period T, which is determined by the expression

$$T=\frac{2}{\beta}\int\limits_{Y_1}^{Y_2}(1+y)^{-1/6}(H-U(y))^{-1/2}dy.$$

We will investigate the permissible values of the total energy H in the case of a fixed value of the speed c. For this it is necessary to consider three possible intervals of variation of the speed c:

$$c^2<c_2^2;\quad c_2^2<c^2<c_1^2;\quad c^2>c_1^2;$$

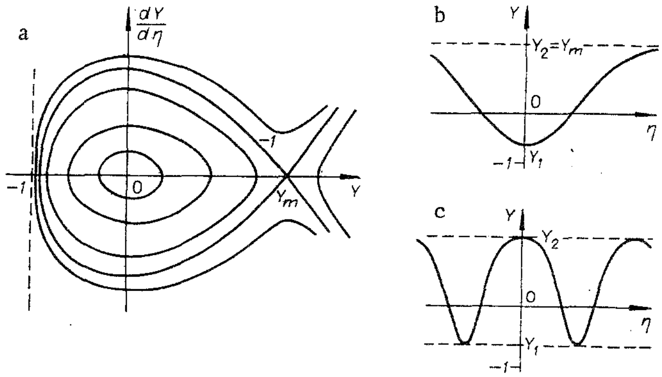

Fig. 3

the corresponding curves 1-3 of the variation of the potential field U(Y) are given in Fig.2; it is obvious that the propagation of a stationary wave in a bubble medium is analogous to the oscillations of a particle in the potential field U(Y) with energy H [4]. In this case, it is necessary for the existence of bounded solutions $Y(\eta)$ that the particle be located in a potential well, i.e., the total energy of the particle should lie between the maximum and minimum values of the potential energy. Consequently, we obtain for the three possible intervals of variation of the speed c three regions of permissible values of the total energy of a bubble H, respectively: $0 \leqslant H \leqslant U_m$; $U_m' \leqslant H \leqslant 0$; and $H > 0$. At the same time, the values Y_m and Y_m', and also U_m and U_m', are determined by the following relations:

$$(1 + Y_m)^{-\gamma} = 1 + [2\alpha/(\gamma - 1)]Y_m, \quad U_m = U(Y_m),$$

and Y_m and Y_m' correspond to a nonzero root of the first equation.

Let us qualitatively consider the types of solutions in each interval of variation of the speed c.

1. Let $c^2 < c_2^2$. The phase trajectories of Eq. (7) are presented in Fig. 3a. Curve 1 ($H = U_m$) is the limiting curve and corresponds to a soliton (Fig. 3b). During the variation of the total bubble energy H from U_m to 0, the phase trajectories are closed (periodic solution, Fig. 3c) and shrink to the center at point (0,0). Upon the variation of the speed c from zero to c_2, the point Y_m tends to zero, and all the phase trajectories shrink to the point (0,0).

Let us derive the constraints on the region of variation of the parameters c^2 and H, which are related to the linearity of the acoustic equations [condition (6)]. Since $Y_1 \leqslant Y \leqslant Y_2$ and $|Y_1| < 1$, $\max|Y| \leqslant \max\{1, |Y_2|\}$. If $|Y|_2 \leqslant 1$, then $z + (1 - z)/\delta \ll 1$ follows from condition (6), and the latter inequality is always satisfied, because $z \ll 1$ and $\delta \gg 1$. When $|Y_2| > 1$ and with the relation $Y_2 < Y_m$ and the estimate $Y_m < -(\gamma - 1)/2\alpha$ taken into account, we obtain

$$c^2 \gg 1/\gamma\delta. \tag{9}$$

Due to the fact that $\gamma\delta \gg 1$, condition (9) does not allow considering waves propagating at low speeds. However, the condition (9) is valid only for a soliton and for the solutions whose phase trajectories are close to the limiting curve (curve 1, Fig. 3a), because Y_1 is of the same order of magnitude as Y_m is for precisely these solutions. If one considers periodic solutions for which $Y_2 \ll Y_m$, then when $c^2 < 1/\gamma\delta$ we obtain the fact that $Y_2 \simeq (1/6)\gamma(\gamma + 1)H$, and its substitution into condition (6) leads to a constraint on the total energy H:

$$H \ll 6/\gamma(1 + \gamma)z.$$

The case of small perturbations $|Y| \ll 1$ is analyzed with the help of the expansion of the function $H - U(Y)$ into a series out to terms of the third order inclusively of (8). In this connection we derive that $Y_m = [2/(1 + \gamma)]D$ and $U_m = Y_m^3/2$. In particular, for a soliton the roots of the function $H - U(Y)$ are expressed in the form $Y_1 = -Y_m/2$, $Y_2 = Y_m$, and the form of the soliton is

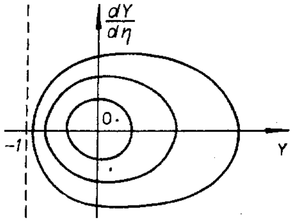

Fig. 4

$$Y(\eta) = Y_m - P/\mathrm{ch}(\sqrt{P}(\beta/2)\eta).$$

The amplitude of the soliton P and its width Δ are determined by the expressions

$$P = -\frac{3}{1+\gamma}D = \frac{3}{1+\gamma}\left(\frac{c_1}{c_2}\right)^2\frac{c^2-c_2^2}{c_1^2-c^2};$$

$$\Delta = \frac{2}{\beta\sqrt{P}}\sqrt{\frac{1-(c/c_1)^2}{(c_2/c)^2-1}}.$$

2. Let $c_2^2 \leqslant c^2 \leqslant c_1^2$; then $U_m' \leqslant H \leqslant 0$. In this case, it is possible to reduce the problem to the one considered for the previous type, using the following substitution:

$$y = \frac{Y-Y_m'}{1+Y_m'};\quad \tilde{c}_2^2 = \frac{c_2^2}{(1+Y_m')^{\gamma+1}+\left[1-(1+Y_m')^{\gamma+1}\right](c_2/c_1)^2};$$

$$\tilde{\beta}^2 = (1+Y_m)^{-(\gamma+2/3)}\beta^2;\quad \tilde{H} = (H-U_m')(1+Y_m')^{\gamma-1}.$$

At this point a transition occurs to a new equilibrium state of the medium:

$$\tilde{V}_0 = V_0(1+Y_m'),\quad \tilde{p}_0 = p_0 + \rho_0 c_0^2 S Y_m',$$

$$\tilde{\rho}_0 = \rho_0\left(1+\frac{S}{c}Y_m'\right).$$

3. Let $c^2 > c_1^2$, then $H > 0$. The phase trajectories of Eq. (7) for some fixed value of the speed c are illustrated in Fig. 4. The point (0,0) is the center. The curves more distant from the center (increase of the wave amplitude) correspond to larger values of the total energy H. As the speed c increases, the phase trajectories become less elongated in the direction of the OY axis. Since all the phase trajectories are closed, only periodic solutions are possible which agree qualitatively with those illustrated in Fig. 3c. The soliton, which propagates with speed $c > c_1$, does not occur. The latter is natural, because when $c > c_1$, the high-frequency harmonics propagate with practically the same speed, and their nonlinear formation cannot be compensated by dephasing due to dispersion as occurred in the region of strong dispersion ($c < c_2$).

We have the following estimate for the roots Y_1 and Y_2 of the function $H - U(Y)$:

$$-\sqrt{[(1+\gamma)/3]H/D} < Y_1 < 0,\ 0 < Y_2 < \sqrt{[(1+\gamma)/3]H/D}.$$

At the same time, we obtain from the condition (6) a constraint on the range of variation of the parameters c and H produced by the linearity of the acoustic equations

$$\frac{\gamma+1}{3}\left(\frac{c_1}{c}\right)^2\frac{Hz}{\delta} \ll c_1^2 - c^2. \tag{10}$$

Condition (10) shows that the total energy and along with it the wave amplitude should be small for speeds c close to the speed c_1 of high-frequency linear waves. One should expect this, since as $c \to c_1$, linear waves are propagated without dispersion and the nonlinearity of the acoustic waves is manifested more strongly.

If we restrict ourselves to small values of the quantity $|Y|$ and use the approximate expression (8) to find the roots Y_1 and Y_2 of the function $H - U(Y)$, then it is possible to produce nonlinear distortion of a small-amplitude harmonic wave. However, one should take special care here, because the formal use of Eq. (8) (quadratic approximation) without taking the smallness of $|Y|$ into account can result in significant distortions of the solution. For example, a false soliton is exhibited when $H = 3[D/(1 + \gamma)]^3$, but in this case $Y_2 = 2D/(1+\gamma) \geqslant (2/3)(c_1/c_2)^2 \sim 1$ and Eq. (8) becomes invalid. In conclusion, explaining the nature of the solutions obtained, we note that the excess pressure p on the right-hand side of Eq. (2) can be expressed with the help of the equation of state in the form of the equation $p = \rho_0 nVc^2c_0^2/[(1 - z)c^2 - 1]$, which results from Eq. (1) for the case of a stationary wave. As is evident, the sign of p/V varies as a function of the relation between c^2 and c_1^2, and when $c^2 > c_1^2$, the excess pressure p increases as V increases, while when $c^2 < c_1^2$, a negative value of p (the pressure decreases) corresponds to an increase in V. The first case corresponds to pulsations of the bubble at frequencies higher than resonance, when the bubble represents a massive impedance and the liquid is elastic; at the same time, an increase in the volume of the bubble is accompanied by compression of the elastic element (a pressure increase). In the opposite case, when $c^2 < c_1^2$, the bubble is the elastic element, and its increase implies a dilatation of the elastic element (i.e., a pressure decrease). We note that the elasticity of the gas in the bubble, which is described by the first term on the right-hand side of Eq. (2), always opposes the expansion, i.e., this term is always negative upon an increase in V. Therefore, when $c^2 < c_1^2$ in the case $(-p) > 0$, the terms on the right-hand side of the equation have unlike signs and compensation of them is possible, which corresponds to the formation of a soliton.

LITERATURE CITED

1. V. E. Nakoryakov, V. V. Sobolev, and I. R. Shreiber, "Long-wavelength perturbations in a gas—liquid mixture," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 5 (1972).
2. E. A. Zabolotskaya and S. I. Soluyan, "Nonlinear propagation of waves in a liquid with uniformly distributed air bubbles," Akust. Zh., 19, No. 5 (1973).
3. B. S. Kogarko, "One-dimensional unsteady motion of a liquid with the onset and development of cavitation," Dokl. Akad. Nauk SSSR, 155, NO. 4 (1964).
4. B. B. Kadomtsev and V. I. Karpman, "Nonlinear waves," Usp. Fiz. Nauk, 103, No. 2 (1971).

ASYMPTOTIC ANALYSIS OF THE PROBLEM OF IGNITION OF REACTIVE MATERIAL BY A HEATED SURFACE

R. S. Burkina and V. N. Vilyunov

UDC 536.46

INTRODUCTION

Due to the Arrhenius dependence of the rate of a chemical reaction on temperature in the statement of many problems of macrokinetics, several relaxation lengths (usually two) are present whose ratio forms a small parameter (for example, the ratio of the chemical reaction and heating zones). Problems of this type pertain to special perturbation problems, for whose solution the method of spliced asymptotic expansions (SAE) is most suitable. The solution of a number of steady-state problems of slow burning and detonation (see [1] and the bibliography in it) has been found with the help of SAE. The attempt to apply SAE to problems of macrokinetics formulated within the framework of partial differential equations* is still very limited [1-3]. Upper and lower limits are found in this paper for the heating time in

*V. S. Berman, "Some problems in the theory of the propagation of a zone with exothermic chemical reactions in gaseous and condensed media." Dissertation in Competition for the Scientific degree of Candidate of Physico-Mathematical Sciences, Institute of the Problems of Mechanics, Academy of Sciences of the USSR, Moscow (1974).

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